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David L. Jaquette and Gary R. Nelson

A Report prepared for

DEFENSE ADVANCED RESEARCH PROJECTS AGENCY

Rand
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PREFACE

This report was prepared as part of Rand's OSD Manpower, Personnel, and Training Program sponsored by the Human Resources Research Office of the Defense Advanced Research Projects Agency (ARPA). With manpower considerations assuming an increasing importance for the Department of Defense, the purpose of this research program is to bring new methodologies to bear to solve present and future military manpower problems.

Under the all-volunteer force, military manpower planners must make several major decisions about the composition of the enlisted force; these decisions concern such issues as the desired ratio of first-term personnel to career force personnel, retention rates for the first-term and career forces, the rate of non-prior-service accessions relative to the size of the force, and the enlisted grade structure and promotion rates. It is well recognized by manpower planners today that these issues are highly interrelated, and to examine these interrelationships, the services have begun to develop a number of manpower models to examine the grade structure, retention rates, and the composition of the force by experience. Less well recognized is the important role played by the complex system of military compensation and retirement in determining the size and composition of the military force. This influence affects accession and retention and can also be seen in the fact that military compensation is a major item in defense costs. The special pay program, through the variable reenlistment bonus and proficiency pay, is used to promote retention in particular occupational areas, but has only recently begun to be tied in through service-wide models to the broad issues of manpower planning.

This report presents a mathematical model of the military manpower system that seeks to determine the optimal distribution of the military enlisted force by years of service. The optimal force is defined as that force which provides the greatest military capability for a given budget cost. Military enlistments and reenlistments depend (among other factors) on military compensation at each point of service. Military compensation rates and, hence, enlistment and reenlistment rates are

chosen to maximize the productivity of the force subject to an annual budget constraint. In a numerical example based hypothetically on the enlisted Air Force, the model indicates the possibility of significant improvements in the productivity of the force through changes in the distribution of personnel by term of service.

Throughout this report, the term "men" is used as a convenience; however, our mathematical model describes the flow of both men and women within the military manpower system.

SUMMARY

Models of the military manpower system have proved to be valuable aids in managing the grade structure and in projecting losses and accessions within different specialties of the services. For the most part, these models have been input-output accounting types, constructed to describe the flow of men among grades and skills by years of service. They were designed to operate within the many institutional constraints of the present system and are not particularly well-suited to optimization of policy alternatives, other than through simulation of military personnel policies.

In this report a mathematical model of military manpower is developed to describe the dynamic flow of personnel within the system, which can then be used to determine optimal military wage rates and lengths of service under steady-state and long-run conditions. This model does not have the detail necessary to supplant existing descriptive flow models but is designed for different purposes. The flows of men within the system reflect enlistments and reenlistments governed by military pay rates. Steady-state pay rates are chosen to maximize the level of military effectiveness subject to an annual budget constraint. Manpower supply is based on enlistment and retention, which are complex functions of the value and timing of wage payments, bonuses, and retirement benefits. The enlistment and retention functions are based as much as possible on extensive previous research on the relationship of military manpower supply to compensation. The level of military effectiveness is a function of the quantities and experience levels of military manpower.

The results obtained in trial runs, which contained both hypothetical data and data for Air Force enlisted personnel, are illustrative of the kinds of results obtainable from such models. The model indicates a steady-state optimum wage policy which yields (1) a higher proportion of the enlisted force in the career force, (2) higher compensation for careerists but lower pay for first-termers, and (3) lower initial accession requirements and lower retention in the third and fourth terms.

The recommendations of the Gates Commission* and the actions taken by the Federal Government to achieve a volunteer force in the short run were to raise first-term pay to increase voluntary enlistments. In contrast, the trial results presented in this report indicate that the long-run all-volunteer force would be most efficiently maintained through improved retention rather than increased enlistments. Since some data used in the example are hypothetical, no recommendations or conclusions as to precise force composition will be made. Future work will attempt to develop better estimates of critical parameters and to extend the force modeling concepts introduced in this report.

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*Thomas Gates, et al., *The Report of the President's Commission on an All-Volunteer Armed Force*, February 1970; and Thomas Gates, et al., *Studies Prepared for the President's Commission on an All-Volunteer Armed Force, Part II, Supply of Personnel to the Military*, Vol. I, November 1970.

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CONTENTS

I. INTRODUCTION

PROBLEMS FACING PLANNERS IN MANNING THE ALL-VOLUNTEER FORCE (AVF)

The AVF has had some obvious effects on the management of military manpower by the services and the Office of the Secretary of Defense. In the area of initial procurement, the influence of the AVF is reflected in higher pay for first-termers, in greater recruiting expense (recruiters and advertising), in changes in the qualitative attributes and motivation* of recruits, and, during part of 1973, in a shortage of recruits, particularly in the Army. The fact that first-term personnel are both more expensive and less plentiful† has strong implications for the composition and structure of the military enlisted force.

The principal issue that manpower planners must face is determining the best, or optimal, structure of the AVF by length of service. This issue is often addressed as the first-term/career ratio, although it actually should encompass the complete distribution of men by year of service. The static composition of the force is an extremely important consideration, because it implies a *retention policy* (the proportion of men continuing from one term of service to the next) and an *accession policy* (the number of enlistments required to maintain a force of given size). The retention policy is important because the normal consequence of changing to an AVF would be an increase in reenlistment rates, at least for first-term personnel. The accession policy may be critical to the success of the AVF where the current need for large numbers of recruits might be eliminated if the force experienced lower turnover rates.

Manpower planners also need to be aware of other issues that are at least potentially related to the structure of the force. First, much of the work in manpower modeling in the past has been concerned

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*Under the AVF the services no longer take in draftees or draft-motivated enlistees; thus, all accessions can be counted as voluntary.

†Under the draft the services could in principle hire an unlimited number of personnel at the prevailing (low) rates of military pay.

with grade management and promotion policies. With limits on the proportion of men serving in the upper grades, a move toward a more experienced enlisted force will reduce promotion opportunities for enlisted personnel. The problem for manpower planners is to establish an equitable and efficient set of promotion policies under the AVF.

Second, one function of the military compensation system is to regulate enlistments and reenlistments. This function has been explicitly recognized, as can be seen in the use of bonuses and proficiency pay to promote accessions and retention in particular occupational areas. Although pay levels are set by Congress and might not be thought of as variable, changes in the structure of the military pay system may be possible as a result of experience with the AVF. Within the past ten years, the following radical changes have, in fact, occurred in the military compensation system: comparability pay increases (1965-1970); special pay programs, such as the variable reenlistment bonus (1966) and the combat enlistment bonus (1972); and the AVF pay increases for first-termers (1969-1971). The first opportunity to consider further large-scale changes in the military pay system, reflecting some experience with the AVF, will be the Third Quadrennial Review of Military Compensation, to be submitted to Congress in 1975.

Finally, a companion issue to that of military compensation is the system of military retirement. Changes in this system are presently under consideration by Congress and proposals for changes have been submitted both by the DOD and by an interagency committee. The retirement system also has an impact on the composition and structure of the military force, which is evident from the high retention rates (near 100 percent) for men approaching 20 years of service and the high loss rates among men who have become eligible for retirement pay. The impact on retention and losses of such changes as a partial vesting of retirement benefits must be considered by military manpower planners.

MANPOWER MODELS

Several military manpower models are in use and under development as force projection and planning aids. For the most part, these are accounting or input-output types of models that determine the number

of men in each year of service, in each military occupational specialty, and in each skill level or grade that will result from implementation of intuitively determined policies on recruitment and advancement. Such static-type models yield the long-run or steady-state force composition resulting from specific planning policies. One such approach is found in the TOPLINE model[(1-4)] which, commencing with the recruited class and using a good* planning policy for promotion, computes the number of men in each rank, occupation, and year of service that results in the long run from implementation of the policy. This policy is then modified to produce a desired force makeup--one that meets DOD institutional and Congressional constraints. The other approach, which is exemplified by TOPCAP,[(2-4)] posits the number of supervisors needed in each occupational group and works backward using good advancement policies and historic retention rates to determine the distribution of men by year of service and skill level required in the stationary or steady-state force.

The dynamic models start with the present-day force makeup and project the force composition forward in time, year by year, using a good promotion policy. This approach permits planners to anticipate surplus or excess personnel in each category so that modification of retention rates and certain skill cross-training can be effected by economic means or by other incentives.†

A third group of planning models uses optimization techniques such as dynamic programming and, more frequently, mathematical programming. Objectives called "goals" are set and penalties are specified for failure to achieve predetermined manpower goals, either in steady-state or in dynamic systems over multiple planning periods. Hiring, firing, and cross-training decision options and their costs are included in these types of models, and a nonlinear mathematical program results, which can sometimes be transformed, allowing the use of conventional linear programming computer programs to find optimal policies.[(6-8)]

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*"Good" in this context is a subjective evaluation made by manpower planners using their various criteria. We later use "good" to describe a particular type of decision policy that is approximately optimal.

†See the Air Force models called Dynamic TOPLINE[(2-4)] and the Navy's ADSTAP projection model PROJECT.[(5)]

Public laws and DOD regulations place a variety of skill, grade, and pay-level constraints on the military's manpower. Planners forced to operate within these constraints are left with basically two means to regulate the manpower distribution. When specific staffing levels are too high, early-out programs and limitation on reenlistments are the most often used control techniques. Yet these are costly manpower management tools, since the alternative of wage reduction would reduce reenlistments at less total cost. Whenever required staffing levels are too low, allocation of the variable reenlistment bonus and proficiency pay is the decision tool used to modify retention rates to attain the desired force composition.

OVERVIEW OF MODEL

The approach taken in the manpower model presented here is to treat military manpower policies as variables to investigate higher-level policy issues related to military compensation, retention, and the rate of initial accessions. Thus, the model can be used to investigate many of the factors that are held constant in other manpower models. This model seeks to determine the optimal composition of the military enlisted force by term of service. The optimal force is defined as that force which provides the greatest military capability for a given budget cost. Since military personnel differ in both productivity and wage rates, differences in the composition of the force affect both military capability and manpower costs.

As a general rule, the most productive force for any given budget cost will result when, for each term of service, the incremental contribution to output just balances the incremental cost of hiring an additional member of that term. This rather simplistic solution, however, is complicated by several factors. First, to retain personnel beyond the first term requires that they enlist initially and that they reenlist the appropriate number of times. Thus, retaining personnel for a fifth term must necessarily reflect five separate enlistment and reenlistment decisions. Second, as numerous empirical studies have shown, enlistment and reenlistment depend upon rates of military compensation.[9-17] Thus the cost associated with either enlisting new

personnel or retaining already-enlisted personnel for an additional term of service is not a constant but depends on the number of enlistments and reenlistments required. Moreover, initial enlistment or reenlistment early in a military career may be affected by compensation that is expected later in the career. For instance, enlistment may be influenced by the presence of a sizable reenlistment bonus. Enlistment and reenlistment are also likely to be affected by the amount of retirement pay. Therefore, the problem of hiring and compensating personnel for a single term of service occurs within a complex system involving enlistment, retention, and military compensation for every term of military service.

The model presented in this report attempts to deal with these relationships. A steady-state solution to the problems is found which chooses rates of military compensation to determine a pattern of enlistment rates. This pattern yields the optimal size and composition of the military enlisted force. An operations research technique called gradient search--a type of nonlinear programming--is used to achieve a computer-based solution to the problem.

This report presents the technical details of the model just described. Section II gives a general mathematical description of the model, Sec. III explains in some detail the assumptions underlying the supply and productivity of military manpower, and Sec. IV presents the results of a numerical example, based hypothetically on the enlisted Air Force. The results indicate the possibility of significant improvements in the productivity of the force through changes in the distribution of personnel by term of service.

II. A MODEL OF PRODUCTIVITY AND COST

The problem addressed in this report is to find rates of military compensation and enlistment and reenlistment which yield a steady-state force of maximum capability for a given manpower budget. The model of the military manpower force emphasizes the productivity of a force of a given size and composition and the cost of hiring such a force. The cost of hiring personnel in a particular term of service is determined by military manpower supply functions which relate enlistment and reenlistment rates to military pay. In applying the model, we are able to draw on research which has estimated supply functions for initial enlistment and first-term reenlistment.[9-17] The productivity of the force is measured by an index which, in the simplest case analyzed, gives productivity weights to men in each term of service. "Optimal" rates of pay are determined by maximizing this productivity index, subject to a budget constraint. These optimal rates of military pay determine the number serving in each term of service under steady-state conditions. The optimization procedure seeks to balance the incremental productivity of a class of personnel with the incremental cost of hiring personnel in that term. The conditions for an optimum require that the ratio of the marginal productivity to the marginal cost be equal for each class of labor. This is a common theorem in the economics of cost and production.

We assume that the total number of men in year of service (YOS) class j ($j = 1, 2, \ldots n$) is characterized by variable x_j. We define the state description vector $x = \{x_j\}$. Similarly, pay vector p is used to describe yearly pay p_j given members of YOS j. The state of the system is thus characterized by vector x, which is in general a function of both pay p and time t and is written $x(p,t)$. The total annual personnel budget is constrained to be less than or equal to B dollars and productivity (military effectiveness) is measured by a function of the number of men in each year of service class and written $S(x)$.

In Appendix B we discuss the mathematical optimization problem, placing it in the context of mathematical decision or control theory,

and describe recent results that justify the use of a simple policy, characterized by an optimal target state. Once the target state x^* is reached, $x(p,t)$ never deviates from x^* under the optimal policy (x^* is called a fixed or stationary point of the process). With these results in mind, we need not even consider x as time variant and need only find the best target or stationary x by minimizing an expression called the average cost criterion* which is cost per unit of time. Therefore, we are able to eliminate the subscript t. We also show how the same optimal pay and force makeup can be found by considering the dual problem of maximizing total productivity per unit of time.

The problem becomes: max $S(x)$, subject to $p'x \leq B$. That is to say, maximize the productivity rate subject to an average expenditure rate constraint. When written as a total Lagrangian with λ expressed in terms of output per dollar, the resulting problem is to find critical points of $L(p,\lambda) = S(x) + \lambda(B - p'x)$ with respect to p and λ with all variables nonnegative. Steady-state force composition described by x is a function of p, $x(p)$.

The general solution yields equations of the form

$$\sum_{i=1}^{n} \frac{\partial x_i}{\partial p_j} (s_i - \lambda p_i) - \lambda x_j = 0 , \qquad \text{for each } j$$

and

$$\sum_{i=1}^{n} p_i x_i = B , \qquad \text{where } s_i = \frac{\partial S(x)}{\partial x_i} .$$

The critical points of L can be obtained by solving n + 1 nonlinear equations for n + 1 unknowns. This is generally difficult unless the functions $x(p)$ and $S(x)$ have a particular form allowing for recursive or iterative methods for solution.

---

*This term has a technical meaning in operations research. It is found as the limit of the actual average of the first N year's costs as N approaches infinity. If C_i is the cost during year i, then the average cost criterion is

$$\lim_{N\to\infty} \sum_1^N C_i/N .$$

We propose two very simple models for illustration. The first model corresponds to an entirely open personnel system, where the employer hires employees of all experience categories on the open labor market. The second model is a strictly closed system, where the employer can obtain experienced personnel only through retention. A generalization of the second model is one that we view as most appropriate to military personnel systems. These gradient search nonlinear programming techniques have been used to solve for the optimal distribution by length of service and rate of pay for enlisted personnel in the Air Force.

MODEL A: AN OPEN LABOR FORCE

Assume that $x_j = x_j(p_j)$, the supply of YOS j is only dependent upon the perceived pay p_j of the same period j. In this kind of system the employer is thought of as hiring each experience class independently on the open market. Retention has no meaning, since the pool of eligibles is so much larger than simply the previous year of service class. From the standpoint of the military, this is an extreme case of lateral entry and is not realistic, since the source of supply is, in fact, the previous year class, x_{j-1}.

In this case, the solution is found by solving

$$\frac{\partial x_j}{\partial p_j}(s_j - \lambda p_j) - \lambda x_j = 0 \qquad \text{for each } j = 1, 2, \ldots n$$

and

$$\sum_{i=1}^{n} p_i x_i = B ,$$

because $\partial x_i/\partial p_j = 0$ for all $i \neq j$. The optimal p_j^* can be found very simply once supply functions $x_j(p_j)$ and $s_j = \partial S(x)/\partial x_j$ are given.

The *jth* condition can also be written

$$\frac{s_j}{p_j - x_j \partial p_j/\partial x_j} = \lambda .$$

The numerator represents the marginal productivity of the *jth* experience class, and the denominator represents the marginal cost of hiring an additional member of the same experience class. Since the ratio of the marginal productivity to marginal cost must equal the Lagrangian multiplier λ, the ratio must be the same for each of the n experience classes hired by the firm. For any two experience classes, the ratio of their marginal products must equal the ratio of their marginal costs for maximum output to be produced for a given budget.

If the employer participates in perfectly competitive labor markets, the marginal cost of an experience class is simply the wage rate. In the competitive case, the marginal productivity of each class is proportional to its wage rate. The factor of proportionality is the Lagrangian multiplier λ, since in the competitive case

$$s_j = \lambda p_j \ .$$

MODEL B: A CLOSED LABOR FORCE

The military is perhaps best represented as a closed system, where no lateral entry is allowed and the supply of experience class j is the product of the supply of experience class j - 1 and a retention rate r_j. The simplest example of supply in our case is to let the retention rate from YOS j - 1 to YOS j be a function only of perceived pay in period j. Thus, $r_j = r_j(p_j)$.

Assume that

$$x_j = r_j(p_j) \cdot x_{j-1} \ ,$$

then

$$x_j = r_1(p_1) \prod_{k=2}^{j} r_k(p_k) \qquad \text{for } j \geq 2 \ .$$

For notational convenience, $x_1 \equiv r_1$, which unlike $r_2 \ldots r_n$ is not bounded by 0 and 1.

The critical point of the total Lagrangian L is found as the roots of the n + 1 equations

$$\frac{dr_j(p_j)}{dp_j} \cdot \left[s_j - \lambda p_j + \sum_{i=j+1}^{n} (s_i - \lambda p_i)\, r_{j+1} \cdots r_i \right] - \lambda r_j(p_j) = 0$$

for all j; $1 \le j \le n$, and

$$r_1 \left[p_1 + \sum_{j=2}^{n} p_j \prod_{2}^{j} r_i \right] = B$$

where again

$$s_j = \frac{\partial S(x)}{\partial x_j} .$$

The solution to this can be obtained by solving recursively for p_n^*, p_{n-1}^*, ... for a given λ. The *jth* equation contains functions $r_j(p_j)$, $r_j(p_{j+1})$, ... $r_n(p_n)$ and is a nonlinear function in only p_j, once p_{j+1}^*, p_{j+2}^* ... p_n^* have been determined. This is done for each of a set of multipliers λ; each λ will generate a particular budget $B(\lambda)$ and search techniques can be used to find the λ^* associated with budget B. These techniques are discussed in Appendix B and applied to find the initial starting or trial wage structure for the model presented in Sec. III.

III. SPECIFIC ASSUMPTIONS ABOUT MANPOWER SUPPLY AND PRODUCTIVITY

To make the model of the closed personnel system explicit, it is necessary to specify the following: (1) military pay received and the pay "perceived" by enlistees and reenlistees, (2) a particular production function, and (3) an explicit form for the enlistment and retention functions r(p). In making these assumptions we move from the area of applied mathematics into the area of the economics of labor supply and production where we must attempt to develop mathematical forms which embody the most realistic economic and behavioral assumptions.

PAY PERCEPTIONS

The military pay perceived by an individual considering enlisting or reenlisting in the military will almost certainly not be limited to the pay rate applying at the time of enlistment or reenlistment. Enlistment contracts for enlisted personnel are for a specified length of time, usually four years in the Air Force and the Navy and three years in the Army. Analysts of military manpower supply have invariably treated military compensation over the entire term of enlistment as the relevant measure of military pay for estimating the supply of enlistees or reenlistees. Based on this information the measure of perception of wage for the *ith* term is the total discounted present value of the military wage and can be written

$$w_i = w_{i1} + \frac{w_{i2}}{1 + s} + \frac{w_{i3}}{(1 + s)^2} + \frac{w_{i4}}{(1 + s)^3} ,$$

where w_{ij} represents total annual pay for the *jth* year of the *ith* term and s is the individual's annual discount rate. The fact that individuals make multiyear commitments to military service also provides a method of simplifying the structure of the model. Instead of using separate year groups, which would require at least 20 separate seniority

classes, we have compressed the time to the four-year increments reflecting the term of enlistment.* The twenty-year military career pursued by the enlisted man is thereby modeled by five four-year terms.†

Our formulation of the military pay perceived by the prospective enlistee or reenlistee is nearly unique in that it includes military pay beyond the term under consideration.‡ Career reenlistment rates in the DOD have always been considerably higher than first-term rates. During periods when retention has been high, these career rates have approached 90 percent. The military retirement system, which provides an annuity for men who have completed twenty years of service, is a major reason for high retention rate in the career force. Since retirement requires a minimum of twenty years of service, the retirement pay is actually "earned" during the fifth four-year term because men who serve only four terms receive no retirement payments under the present system. To reflect future pay, such as retirement, in the enlistment and reenlistment decision, we define an expected military earnings stream which includes pay beyond a single term of service. This future pay is reduced both by the discount factor d reflecting the rate of time preference and the subjective probability of remaining in the force until that time. (The discount factor is based on a four-year term: $d = 1/(1 + s)^4$. To simplify this assumption mathematically, we assume that all potential enlistees or reenlistees foresee the same retention probability and that this subjective probability is the actual retention rate r. The value of the perceived military earnings stream can be written for a two-period model as

$$p_1' = w_1 + w_2 r_2 d_2 ,$$

---

*Since our analysis is steady state, this compression of time has no effect on the solution of the model.

†There is no mathematical necessity that the terms be equal in length. Hence, the model can be applied to the Army where the first term is usually three years and where later terms are often six years.

‡This suggestion, however, has been made by a number of other analysts at Rand, including W. H. Albright, B. Dieffenbach, and G. Gotz.

where w_1 and w_2 are pay rates for the first and second terms, r_2 is the retention rate between the first and second terms, and d_2 is the (multi-year) discount factor applicable to second-term pay.

One problem with this formulation is that even if first- and second-term pay are equal, the present value will change as the variable r_2 changes. This will not occur if pay is expressed in annual pay equivalents through the use of an annuity factor:

$$p_1 = \frac{p_1'}{1 + r_2 d_2} = \frac{w_1 + w_2 r_2 d_2}{1 + r_2 d_2} .$$

The use of this factor results in a perceived pay whose units are dollars per year (or per term). This annual rate is a weighted average of the annual rates, w_1, w_2, ... w_5. If retention probability is high, heavy weight is given to such future pay as retirement. If retention probability is zero, all weight is given to pay in the term under consideration. Obviously, there is a simultaneous relationship at work. Retention rates offset the valuation of future pay, but future pay has, of course, an effect on retention. Through the optimization procedure* we arrive at an evaluation of perceived pay as well as the determination of optimal retention rates.

Table 1 presents the algebraic values of perceived pay as a function of actual pay rates, w_1, ..., w_5 and the lump-sum value of retirement pay R,* henceforth always included in w_5. The model searches for optimal values of perceived pay p_i and the resultant reenlistment rates $r_i(p_i)$. It is possible to specify uniquely the actual pay rates through a linear transformation of the p_i into w_i. These equations also appear in Table 1. The model, as written, is general enough to allow different discount rates for different points in the military career. This corresponds to assumptions made by military manpower supply analysts, in which first-termers are accorded a higher rate of time preference (30 percent per year) than men already in the force (10 percent and 5 percent per year).†

*In this simplified model we assume that all men retire at exactly twenty years of service.

†See, for instance, Refs. 9 and 12.

Table 1

RELATIONSHIP BETWEEN PERCEIVED PAY AND ACTUAL ANNUAL WAGES

A. Perceived pay (p_i) as a function of actual annual pay (w_i) and lump-sum value of retirement (R) which is included in wages paid in the fifth term, w_5

$$p_5 = w_5$$

$$p_4 = \frac{w_4 + w_5 r_5 d_5}{1 + r_5 d_5}$$

$$p_3 = \frac{w_3 + w_4 r_4 d_4 + w_5 r_4 r_5 d_4 d_5}{1 + r_4 d_4 + r_4 r_5 d_4 d_5}$$

$$p_2 = \frac{w_2 + w_3 r_3 d_3 + w_4 r_3 r_4 d_3 d_4 + w_5 r_3 r_4 r_5 d_3 d_4 d_5}{1 + r_3 d_3 + r_3 r_4 d_3 d_4 + r_3 r_4 r_5 d_3 d_4 d_5}$$

$$p_1 = \frac{w_1 + w_2 r_2 d_2 + w_3 r_2 r_3 d_2 d_3 + w_4 r_2 r_3 r_4 d_2 d_3 d_4 + w_5 r_2 r_3 r_4 r_5 d_2 d_3 d_4 d_5}{1 + r_2 d_2 + r_2 r_3 d_2 d_3 + r_2 r_3 r_4 d_2 d_3 d_4 + r_2 r_3 r_4 r_5 d_2 d_3 d_4 d_5}$$

B. Actual pay (w_i) as a function of perceived pay (p_i)

$$w_5 = p_5$$

$$w_4 = p_4(1 + r_5 d_5) - p_5 r_5 d_5$$

$$w_3 = p_3(1 + r_4 d_4 + r_4 r_5 d_4 d_5) - p_4(r_4 d_4 + r_4 r_5 d_4 d_5)$$

$$w_2 = p_2(1 + r_3 d_3 + r_3 r_4 d_3 d_4 + r_3 r_4 r_5 d_3 d_4 d_5)$$
$$- p_4(r_3 d_3 + r_3 r_4 d_3 d_4 + r_3 r_4 r_5 d_3 d_4 d_5)$$

$$w_1 = p_1(1 + r_2 d_2 + r_2 r_3 d_2 d_3 + r_2 r_3 r_4 d_2 d_3 d_4 + r_2 r_3 r_4 r_5 d_2 d_3 d_4 d_5)$$
$$- p_2(r_2 d_2 + r_2 r_3 d_2 d_3 + r_2 r_3 r_4 d_2 d_3 d_4 + r_2 r_3 r_4 r_5 d_2 d_3 d_4 d_5)$$

ENLISTMENT AND REENLISTMENT FUNCTIONS

In manpower systems without lateral entry, the supply of manpower for the *ith* term is equal to the number of enlistees in term i - 1 times the subsequent retention rate. Thus, for the highest term

$$x_5 = x_1 r_2 r_3 r_4 r_5 ,$$

and in general

$$x_i = x_1 r_2 r_3 \ldots r_i .$$

Each of these factors itself constitutes a supply function. We define $x_1 = r_1$ to simplify notation. Each supply function is assumed to depend only on the annual pay perceived by potential enlistees and reenlistees. Although civilian earnings and tastes for military service obviously influence military manpower supply, these are not variables within the context of the model but rather are included indirectly in parameters of the retention functions.* Thus, we assume

$$r_i = r_i(p_i) .$$

We have adopted the function

$$r(p) = ce^{-ap^{-b}}$$

as the form for each of the enlistment and reenlistment functions. This exponential function is a positively sloped s-shaped function, and if c is equal to 1.0, it is bounded by 0 and 1 for all positive levels of perceived pay. (Examples plotting r(p) for several sets of parameter values are shown in Fig. 1.) This guarantees, for instance, that reenlistment rates will always fall between 0 and 1.0 regardless of the level of military pay. The parameter c for the reenlistment functions

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*It is possible that retention in some particular specialty will influence civilian wage rates in some set of similar occupations, but we abstract from these complexities.

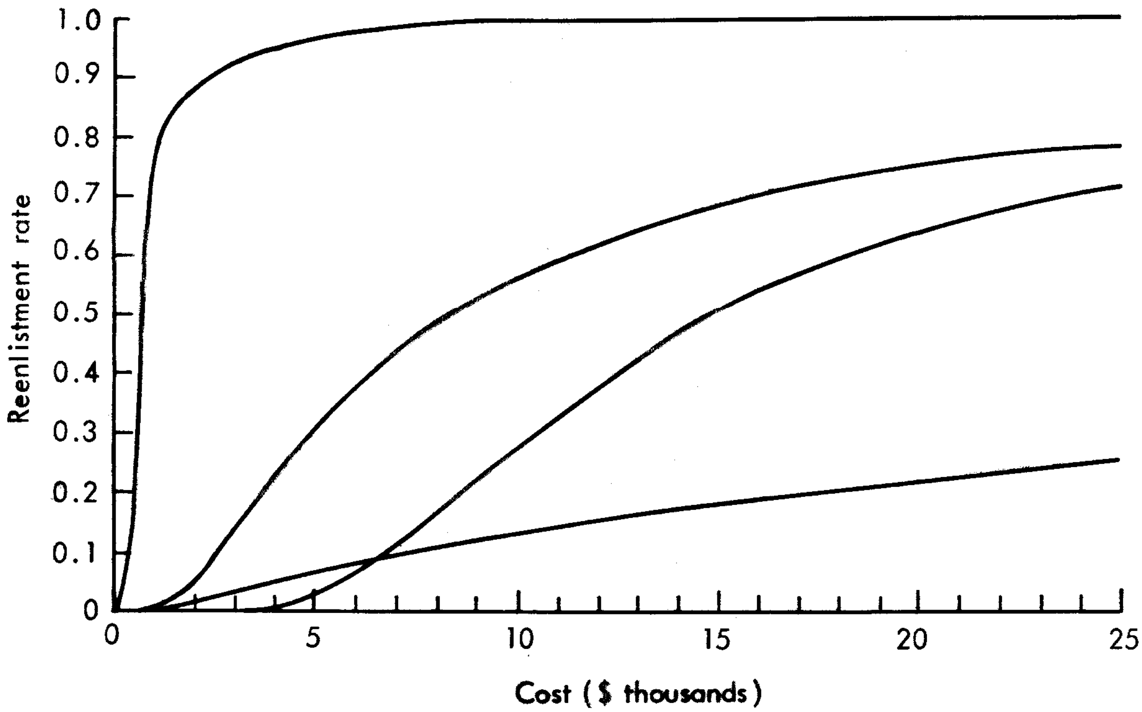

Fig. 1 — Retention rates as a function of pay for various a and b parameter values ($r(p) = c \cdot \exp(-a \cdot p^{-b})$)

$r_2, \ldots, r_5$ serves two purposes: It reflects the proportion of men in the previous term eligible to reenlist, and it reflects attrition of men who do reenlist. For the special case of r_1, which represents the number of enlistees rather than an enlistment rate, c_1 serves a different purpose from $c_2, \ldots, c_5$. The number of men eligible to enlist in the population as a whole is reflected by c_1. Since this is a steady-state model with a four-year time period, the eligible cohort is the four-year age cohort from which eligibles are drawn in the civilian population--approximately five million men.

Although this exponential function has not (to our knowledge) been used for supply analysis in the past, it is similar to the log-normal function and a class of logistic (or logit) functions, which have been applied.*

*See, for instance, Refs. 10, 13, and 17.

Log-normal: $$r(p) = \int_0^p \frac{1}{\sqrt{2\pi}\sigma} e^{-(\ell n z - u)^2/2\sigma^2} dz .$$

Logistic: $$r(p) = \frac{\exp(a + b \ \ell n \ p)}{1 + \exp(a + b \ \ell n \ p)} .$$

The attractiveness of the exponential function lies in both its mathematical tractability and the reasonableness of its shape. The derivative of reenlistments with respect to perceived pay is

$$\frac{dr}{dp} = abp^{-b-1} ce^{-ap^{-b}} = abp^{-b-1} r .$$

The elasticity of supply is

$$\frac{p}{r}\frac{dr}{dp} = abp^{-b} .$$

This formulation greatly simplifies the difficult problem of finding a solution to the set of nonlinear first-order conditions which are needed in any optimization.

MILITARY PRODUCTIVITY INDEX

Military capability or effectiveness can be formulated as a function of inputs of capital and labor. In this model, which is devoted to the composition of military manpower, all other factors of production, such as various types of capital equipment and categories of civilian labor, are treated as constants. The principal issue is the degree of substitutability between various military experience categories--in this case the tradeoff in terms of effectiveness between men in different terms of service. In this regard we investigate two production functions: the linear and the multiplicative (Cobb-Douglas) production functions. In vector notation the linear production function is written

$$S(x) = s'x = \sum_{j=1}^{n} s_j x_j .$$

Where the contributions of the different year classes are additive, the elasticity of substitution* between different year classes is infinite. In a sense, the different year classes are perfect substitutes for one another, in that tradeoffs (even unlimited) can be made between x_i and x_j at the ratio s_j/s_i without any sacrifice in productivity. Since s_i and s_j are unlikely to be equal, perfect substitutability does not imply that less-experienced personnel have marginal productivity equal to more experienced personnel. Although this production function represents a polar extreme, it is at least partially supported by empirical research[18] which shows elasticities of substitution between entirely different classes of labor to be as high as 10. In any case the first-order changes in output can always be modeled with a linear production function.

In the Cobb-Douglas production function

$$S(x) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_5^{\alpha_5} .$$

The sum of the parameters $\sum \alpha_i$ is a measure of the returns to scale and is arbitrarily set equal to 1.0, since results of our model are independent of this normalization. Furthermore, in this function the elasticity of substitution is 1.0. In the Cobb-Douglas some quantity of input from each labor class is required to produce any level of output.

In the linear function the marginal productivity of any year class is

$$\frac{\partial S}{\partial x_i} = s_i ,$$

---

*The elasticity of substitution is defined as the elasticity of the ratio of the marginal products of two factors or production to changes in the ratios of the two factors. If y represents the ratio of the marginal productivity of x_1 and x_2 and z is the ratio x_1/x_2, then elasticity of substitution is defined as zdy/ydz, where total output remains constant.

which as we have noted is independent of other labor and capital inputs. In the Cobb-Douglas, marginal productivity is

$$\frac{\partial S}{\partial x_i} = \alpha_i \frac{S(x)}{x_i}$$

and depends on the ratio of total labor productivity to the inputs from a given year class. As the quantity x_i increases, marginal productivity declines. This is a classical case of diminishing marginal returns to any factor of production. The ratio of the marginal productivities of two factors is

$$\frac{\partial S/\partial x_i}{\partial S/\partial x_j} = \alpha_i x_j / \alpha_j x_i \ .$$

The ratio is inversely related to the ratio of the two factors of production. The difference between the linear and Cobb-Douglas functions can be further examined by looking at the various combinations of x_i and x_j required to produce a given level of output. This appears in Fig. 2.

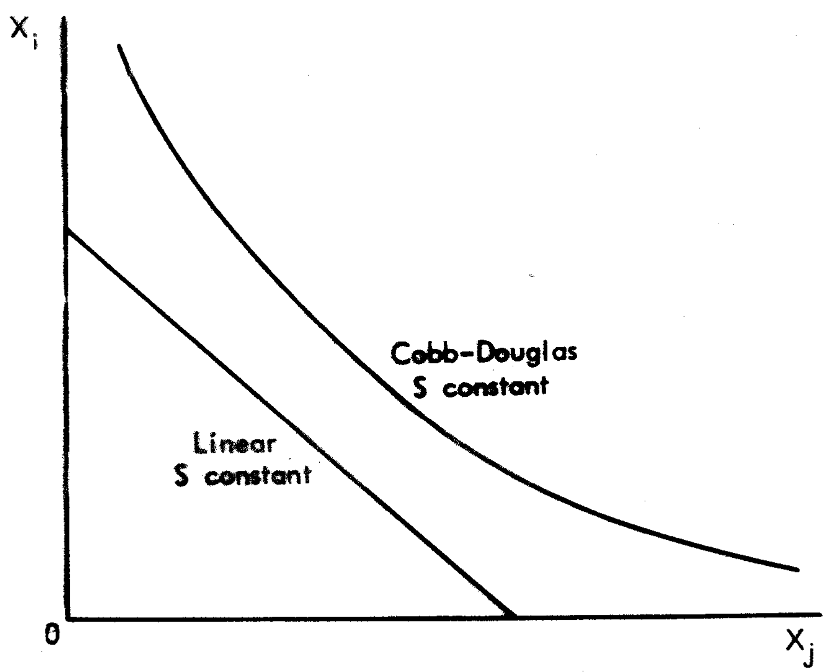

Fig. 2 — Production isoquants

The Cobb-Douglas production isoquant asymptotically approaches each axis, signifying that some of each factor is necessary to the production process. In the linear case the isoquant is a straight line and the same level of output can be produced using x_i or x_j exclusively or by using some linear combination of the two.

BUDGET CONSTRAINT

The annual steady-state wage cost to the Federal Government is the product of the annual wage rate vector w and the manpower vector x: $x'w$. This is called the budget and equals $\sum_{i=1}^{5} x_i w_i$. Wage rate w_5 is defined to include retirement costs.* The objective is to maximize the productivity index subject to a budget constraint

$$x'w \leq B \text{ .}$$

Since the model is formulated in terms of perceived pay p, we must transform w into p via the equations contained in Table 1. In matrix terms we can write perceived pay as a function of w: $p = Aw$. The elements of A, which are nonzero only above the diagonal, are

$$A_{i,i+j} = \frac{r_{i+1} \cdots r_{i+j} \, d_{i+1} \cdots d_{i+j}}{1 + r_{i+1} \, d_{i+1} + \cdots + r_{i+1} \cdots r_5 d_{i+1} \cdots d_5} \text{ .}$$

An inverse matrix A^{-1} can be used to write w as a function of p: $w = A^{-1}p$. More detail is given in Appendix B.

Differentiation of the total Lagrangian L with respect to decision variables p produces six nonlinear simultaneous equations. Gradient search techniques have been applied to this problem and are discussed in Appendix B. This technique has been programmed for the computer and employed to find the optimal p maximizing productivity subject to the wage constraint for both cases of this model. The linear transformation represented by matrix A^{-1} can be used along with the pay vector determined optimal within the model to reconstruct the average wages to be paid to members of each term of service.

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*In reality, of course, military retirement costs are not included in the services' budget.

IV. NUMERICAL RESULTS

The principal source of existing data for developing numerical results from the model is the considerable econometric research into the supply of military personnel. The Gates Commission reported a DOD-wide pay elasticity of supply for enlistees of 1.25, a figure which approximated that reported by studies of the response of enlisted personnel to the AVF pay increases.(13) It also reported on elasticity of supply for first-term reenlistees of 2.36 for the Air Force and similar values for the other services. The derivation of supply functions from these point estimates of elasticities is discussed in detail in Appendix A. The basic process was to assume that these elasticities held for the enlistment and first-term reenlistment rates observed in the Air Force for fiscal 1972. Since the exponential supply function has a variable elasticity, the point elasticity depends on the actual rate of enlistments and reenlistments. The econometric studies which yielded elasticities of 1.25 and 2.36 were based on only a single term of military pay. Elasticities with respect to total perceived pay are greater than elasticities with respect to pay for a single term of service. Reenlistment supply functions used for the second, third, and fourth reenlistment decisions are based wholly on assumption, in the absence of empirical evidence, and exhibit elasticities far below those shown by the first-term reenlistment supply. Low elasticities at present career retention rates can be expected to occur, since the potential for increases in retention is extremely limited for rates already in excess of 90 percent.

Production function S(x) was estimated using preliminary estimates obtained from an on-the-job training study under way at Rand.(19) For the linear production model, this provided us with an estimate that first-term average productivity was 0.68 compared with productivity of fully trained journeymen. During the second term, the average productivity was assumed to be 1.0. Subsequent productivity was obtained by viewing civilian alternative wage rates of workers in similar fields and background by number of years on the job. Full data were not

available for this study but Becker[20] has provided rough estimates of civilian wage increases by age, which when converted to term of service were 1.28, 1.513, and 1.755. We assumed that productivity of civilian workers is about the same as that of military workers of the same age.

For the Cobb-Douglas production function, the same tradeoffs between experience classes were assumed valid at the current (1972) composition of the force. Thus, for example,

$$\frac{\partial S/\partial x_1}{\partial S/\partial x_2} = \frac{\alpha_1 x_2}{\alpha_2 x_1} = 0.68 \ .$$

Similarly,

$$\frac{\alpha_3 x_2}{\alpha_2 x_3} = 1.28 \ ,$$

$$\frac{\alpha_4 x_2}{\alpha_2 x_4} = 1.513 \ ,$$

$$\frac{\alpha_5 x_2}{\alpha_2 x_5} = 1.755 \ .$$

Knowing retention rates and the fact that $\sum \alpha_i = 1$ enables one to specify completely the Cobb-Douglas production function from the same marginal productivity assumptions used for the linear case.

Discount factors for first term were constructed using an interest rate of 30 percent per year ($d_2 = (1/1.30)^4 = 0.350$). During the second term, the time value of money was calculated using 10 percent per year ($d_3 = 0.683$). Discount for subsequent terms and for retirement is based on 5 percent per year (d_4, $d_5 = 0.823$). Precedence for using these values can be found in Refs. 9 and 12.

Productivity, current wages, current retention, and perceived pay generated using the appropriate discount factors and retention rates are shown in Table 2. The table also shows the optimal values of wage, retention, and perceived pay for the linear and for the Cobb-Douglas production functions.

Table 2

Current and Optimal Pay Rates and Manpower Distribution

Item	Term of Service 1	2	3	4	5	Total
Current (Transient) Manpower (1972)						
Average wage, $	6,110	8,130	8,040	8,630	22,030[a]	$5.174 billion
Perceived pay, $	6,840	10,210	11,870	14,600	22,030	
Gross retention[b]	0.20	0.273	0.765	0.951	0.973	
Number of men	306,100	68,600	47,200	45,000	89,800	556,700
Productivity	0.68	1.0	1.28	1.513	1.755	562,400 units
Optimal Steady State Manpower Plan Using Linear Production Function						
Average wage, $	3,100	11,300	10,940	10,700	16,300	$5.174 billion
Perceived pay, $	5,590	11,760	12,140	13,060	16,300	
Gross retention	0.137	0.518	0.785	0.910	0.888	
Number of men	227,800	118,000	92,600	84,300	74,900	597,600
Productivity	0.68	1.0	1.28	1.513	1.755	650,100 units
Optimal Steady State Manpower Plan Using Cobb-Douglas Production Function						
Average wage, $	4,430	10,220	10,300	10,390	16,500	$5.174 billion
Perceived pay, $	6,060	11,090	11,810	12,980	16,500	
Gross retention	0.161	0.422	0.760	0.907	0.895	
Number of men	267,500	112,800	85,700	77,700	69,500	613,200
Productivity coefficient	0.3657	0.1468	0.1437	0.1616	0.1819	

[a]Wage in the fifth term consists of current $9,230 average wage plus $12,800 annual equivalent for the retirement component.

[b]Gross retention = x_{j+1}/x_j uncorrected with c_j.

Currently first-termers represent 55 percent of the USAF enlisted force with an enlistment rate of 76,500 per year. In the optimal schedule for linear production this figure is reduced to 38 percent with enlistments of 56,800 required annually. Total productivity increased

16 percent, and total number of men in service increased by 7.35 percent. These changes would leave the annual budget unchanged at $5.174 billion.

Analysis of Table 2 shows that under a Cobb-Douglas production function the optimal force composition would consist of 44 percent first-termers requiring 65,800 recruits annually. Reenlistment rate for first-termers is again substantially higher than presently found in the Air Force.

The slightly irregular pattern of military pay reported under the optimal results is not as troublesome as it may appear. In fact, military pay in any term can be thought of as the sum of regular compensation, bonuses, and retirement vesting. Under the present system, of course, reenlistment bonuses occur mainly in the second term and retirement vesting occurs at the twenty-year point or the fifth four-year term.

Table 3 indicates a wage, bonus, and retirement vesting program that achieves this optimal pay schedule while providing a smooth and increasing stream of regular compensation. A study is needed to determine the best combination and the particular case shown is only an example.

Table 3

WAGES, BONUS, AND RETIREMENT

(In dollars)

Term	Optimal Wages	Annual Basic Pay	Reenlistment Bonus	Annual Retirement Vesting
1	3,100	3,100	0	0
2	11,300	6,000	4,000	1,300
3	10,940	8,000	1,500	1,440
4	10,700	9,000	0	1,700
5	16,300	10,000	0	6,300

NOTE: All figures are in annual equivalents.

By way of analysis of the results so far obtained, it does appear that the higher first-term reenlistments combined with lower career force reenlistment rates as a whole are desirable. The effect of learning the job, as indicated by low productivity during the first term, is the primary factor. With greater retention, a reduction in the number of enlistees required to maintain the staffing levels and output is feasible.

Sensitivity analysis of the results of Table 2 to changes in productivity and supply reenlistment parameters was conducted. While actual optimal wage and retention rates varied as expected, the qualitative conclusions persisted that a decrease in the number of first-termers (accomplished by reducing their pay) and an increase in retention of these first-termers (accomplished by increasing pay) were both desirable.

Furthermore, retention at the end of the third and fourth terms was somewhat reduced by slight reduction in pay. We postulated in Table 3 that an early retirement vesting system could be useful to (a) smooth out the optimal wage fluctuations required to achieve optimal retention and (b) make early-out options more equitable and appealing. Further study of this proposal and its effect on retention seems advisable.

V. CONCLUSIONS

At this point in the study, conclusions for extensive policy changes are obviously premature. The purpose of this research has been to demonstrate the usefulness of the approach and verify the model. Further work must be undertaken to refine the rough estimates of retention parameters and productivity before recommendations would be warranted. Also, as we have indicated, it would be desirable to reimpose on this system some of the present legal and institutional constraints. This will permit us to examine the cost of such constraints.

The problem of determining the optimal length of the first enlistment (and perhaps all reenlistment terms) is another complex one, since little experience is available on enlistment or reenlistment behavior as a function of term length.* If this empirical difficulty can be overcome, the problem becomes one of discrete optimization; that is, one plugs in a different plan of enlistment terms and runs the model to determine if total productivity has increased with the same budget. We then choose the best plan for term of service length.

Our model can be applied to subdivisions of the service such as a single military occupation specialty. Here productivity and reenlistment and supply parameters might more easily be measured.

Work currently being done at Rand covers a variety of manpower topics, the results of which will aid in determination of productivity and the effect of pay increases on retention. Modification of the model as needed and improved estimation of parameters will be made as this work progresses.

Once a steady-state optimal policy and target states are known, the next question is how to achieve them. This problem has not been addressed here. Preliminary work suggests that a transient policy might be found using dynamic or mathematical programming and furthermore that the policy is highly dependent upon the interest rate. We are able to

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*The Army Reserve has recently begun experimentation with reserve enlistment terms of various lengths.

construct good* policies, in the same sense that Flynn[21] describes them. This type of transient policy simply guarantees reaching the optimal target state in a finite number of steps (in our case, n periods), not necessarily in an optimal way.

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*"Good," as used here, has a technical meaning. See Appendix B.

Appendix A

EMPIRICAL ESTIMATES OF SUPPLY FUNCTIONS

In this appendix, a method is discussed for estimating parameters, a, b, and c of the reenlistment function r as a function of pay p. A minimum of three data points for pay and retention rate are needed to obtain estimates of these three parameters. Only one point, that point being current pay and current retention rate, is available.

One approach is to design a controlled experiment where pay is reduced for one group, increased for another, and left the same for the remainder. Three data points generated would make simple estimation of a, b, and c possible.

The other option is to make use of outside information. Data are available on the percentage eligible for reenlistment at each decision point. A man may be ineligible to reenlist for medical, psychological, or disciplinary reasons. The remaining fraction, the maximum reenlistment achievable, is parameter c.

Another bit of outside information comes from previous studies[9, 11-13,15,22,23] that have been concerned with determining supply elasticities at initial enlistment and first-term reenlistment decision points. The Gates Commission used an enlistment elasticity of 1.25 and a first-term reenlistment elasticity of 2.36. Currently no estimates have been made of second, third, and fourth reenlistment supply elasticities. However, intuitive assumptions have been made here to provide the complete parameter set. The assumption was made that the curve for reenlistment in term 3 would pass through 99 percent at \$20,000, the curve for term 4 through 60 percent at \$10,000, and the curve for term 5 through 50 percent at \$12,000. As the purpose of this research is to demonstrate the model and not at the moment to provide concrete answers for policy change, at this point this guess seems reasonable. Table A-1 displays the input used.

Table A-1

MODEL PARAMETER VALUES

Period	Data							Discount Factors[e] $d = (1 + i)^{-4}$
	r[a]	p[b]	η_p	η_w[c]	a	b	c[d]	
1	0.20	6,838	1.70	1.25	1.41×10^4	1.03	$5 \times 10^6/3$	
2	0.321	10,215	6.70	2.36	4.95×10^{23}	5.90	0.85	0.350
3	0.805	11,874	1.28	0.359	2.20×10^{23}	5.89	0.95	0.683
4	0.961	14,596	0.269	0.0882	5.11×10^{26}	6.75	0.99	0.823
5	0.983	22,026	0.106	0.0445	3.40×10^{24}	6.05	0.99	0.823

NOTE: Supply function, $r = ce^{-ap^{-b}}$; supply elasticity to perceived pay, $\eta = abp^{-b}$.

[a]Reenlistment rate r among eligibles, gross reenlistment is $c \cdot r$.

[b]Perceived pay calculated from average current wage, reenlistment rates, and discount factor. Retirement of $3,700/year average is discounted at 5 percent over 28 years. This has present value of about $52,000. This adds approximately $12,800 to wage in each year of the fifth term.

[c]Elasticity to perceived pay changes is always greater than elasticity to wage changes because wage changes occur over the four-year term. Authors of the Gates Commission study estimate percentage change in reenlistment/percentage change in wage. This elasticity, indicated here as η_w, is found to be related to η_p by

$$\eta_p = \frac{E}{W} \eta_w,$$

where E is the total expected present worth of all future military payments, since

$$\eta_w = \frac{w}{r} \frac{\partial r}{\partial w} = \frac{\partial p}{\partial w} \frac{w}{p} [p \frac{\partial r}{\partial p}] = \frac{\partial p}{\partial w} \frac{w}{p} \eta_p = \frac{w}{\text{present worth}} \eta_p \ .$$

[d]Parameter c_1 is distinct in that it indicates the maximum size of the pool of men who could possibly enlist. Of the overall pool of eight million men (every four years) only 60 percent are fit for service. The USAF gets approximately one-third of those enlisting in any service. The remaining parameters, c, are reasonable guesses based on estimates of medical and social problems.

[e]Interest rates of 30 percent, 10 percent, 5 percent, 5 percent, and 5 percent were used following Refs. 9 and 12, to calculate discount factors.

Appendix B

MATHEMATICAL OPTIMIZATION

This appendix presents an outline of the mathematical model and optimization problem and a discussion of the computational methods used to obtain the optimal solution. The general problem of choosing management decisions is a classical problem of operations research. Similar systems in engineering and mathematics are studied by control theorists and applied mathematicians.

Discrete dynamic programming or Markov decision theory is applicable to a subset of these general problems when the future behavior of the system can be predicted on the basis of the current state, time, and decision but independent of the path (sequence of states, times, and decisions) taken to get to the present. This theory is concerned with the optimal sequential control of a periodically observed and controlled stochastic process.

Many researchers have determined the form of the optimal policy for specific models that are specifically restricted subsets of the general Markov decision process. The construction of optimal policies for these models is often an even more difficult problem. Fortunately, policies can be found using construction algorithms developed by these same researchers for some of these models.

Flynn[21] considers a deterministic Markov decision model with direct application to the military manpower system. This system consists of productive units that age and possibly leave the system early or ultimately retire. Through an enlistment policy which recruits the youngest productive units and through improved retention by increasing salary, the productivity rate can be maintained constant. The production rate is a linear function of the number of units in the different age groups. The general decision on wage and recruitment is allowed to be a function of both time and the current state of the system. An optimal policy in this model is one which minimizes the total present worth of all payments subject to the productivity rate constraint.

In this paper, Flynn shows that minimizing the average cost (total average cost per unit of time) as an alternative to the total present worth

criterion will produce a good target state (long-run manpower force distribution). This target state is the same stationary point that the optimal policy for the total present worth criterion would ultimately reach, and a "steady-state" policy can be constructed to reach this target state from the initial system state. The present worth cost criterion differs from the optimal cost by an amount whose bound is independent of the interest rate. Thus, while a truly optimal policy is almost impossible to calculate, a good one can be easily found by minimizing the average cost criterion.

These results can be used to justify the use of the long-run policy found in this report for the dual problem of maximizing the average productivity rate of our linear production model subject to a constraint on annual costs. If this maximum productivity is chosen as the value of the productivity constraint imposed by Flynn, then the target state and wage decisions are identical and this average cost criterion will be minimized and will equal our annual budget. Thus the model we present here determines a decision and target state that approaches the optimal decision.

Using dynamic programming to directly determine an optimal program is not feasible because of the large number of states and actions of manpower planning models; however, direct application of dynamic programming could prove useful for small-scale problems where time dependent solutions are desired.

The linear production model uses x as a state vector that describes the number of men in each term of service. Since we are looking for a good target state, x^*, we drop the dependence on time. Assume vector s defines the productivity per man in each term of service. A distinction is made between perceived pay and wage as described in the main text. The problem is, maximize $x's$ subject to $x'w \leq B$, where $p = Aw$ and with

$$x(p) = \begin{pmatrix} r_1(p_1) \\ r_1(p_1)\ r_2(p_2) \\ r_1(p_1)\ r_2(p_2)\ r_3(p_3) \\ \cdot \\ r_1(p_1)\ \ldots\ldots\ r_5(p_5) \end{pmatrix} .$$

Since p is the decision variable driving state vector x, we can transform the problem to one involving p alone: maximize x's subject to $x'A^{-1}p \le B$, where

$$A^{-1} = \begin{bmatrix} 1 + r_2d_2 + r_2r_3d_2d_3 \cdots \ - \cdots \\ \quad + r_2d_2 \cdots r_5d_5 & 0 & 0 & 0 \\ 0 & 1 + \cdots & \begin{matrix} - r_3d_3 - r_3r_4d_3d_4 \\ r_3d_3r_4d_4r_5d_5 \end{matrix} & 0 & 0 \\ 0 & 0 & 1 + r_4d_4 + r_4r_5d_4d_5 & - r_4d_4 - r_4r_5d_4d_5 & 0 \\ 0 & 0 & 0 & 1 + r_5d_5 & - r_5d_5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The total Lagrangian is $L = s'x - \lambda[x'A^{-1}p - B]$.

The term $x'A^{-1}p$ is a scalar and when expanded can be reordered. This reordering indicates that the same scalar could be constructed from $x'Cp$, where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ d_2 & 1 - d_2 & 0 & 0 & 0 \\ d_2d_3 & d_3(1 - d_2) & 0 & 0 & 0 \\ d_2d_3d_4 & d_3d_4(1 - d_2) & d_4(1 - d_3) & 1 - d_4 & 0 \\ d_2d_3d_4d_5 & d_3d_4d_5(1 - d_2) & d_4d_5(1 - d_3) & d_5(1 - d_4) & 1 - d_5 \end{bmatrix}.$$

Taking the derivative of L with respect to p to obtain the first-order conditions for critical points is easier using this matrix, C, which is independent of p.

$$\frac{\partial L}{\partial p} = \frac{\partial x'}{\partial p} \cdot [s - \lambda Cp] - \lambda C'x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\frac{\partial L}{\partial \lambda} = B - x'Cp = 0 \ .$$

We define

$$M \equiv \frac{\partial x'}{\partial p} = \begin{pmatrix} \frac{\partial r_1}{\partial p_1} & \frac{\partial r_1 r_2}{\partial p_1} & \cdots & \frac{\partial r_1 r_2 r_3 r_4 r_5}{\partial p_1} \\ \vdots & & & \\ \frac{\partial r_1}{\partial p_5} , & \cdots & & \frac{\partial r_1 r_2 r_3 r_4 r_5}{\partial p_5} \end{pmatrix} .$$

In our model

$$M = \begin{Bmatrix} a_1 b_1 p_1^{-b_1 - 1} \cdot (r_1, r_1 r_2, \ r_1 r_2 r_3, \ \ldots, \ r_1 r_2 r_3 r_4 r_5) \\ a_2 b_2 p_2^{-b_2 - 1} \cdot \ (0, r_1 r_2, \ r_1 r_2 r_3, \ \ . \ , \ \ . \ \ ) \\ \vdots \qquad\qquad \vdots \\ a_5 b_5 p_5^{-b_5 - 1} \cdot \ (0, \ \ 0, \ \ \ 0, \ \ 0, \ r_1 r_2 r_3 r_4 r_5) \end{Bmatrix} .$$

For the case of Cobb-Douglas production function, Flynn's analysis may not apply. Theoretical work is needed before we could state that our steady-state approach has produced a good policy in the same sense.

The steady-state optimal policy for the Cobb-Douglas production function can be found by maximizing the revised Lagrangian

$$L = \sum_{i=1}^{5} \alpha_i \ \ell n \ x_i - \lambda [x' A^{-1} \ p - B] \ .$$

The first-order conditions are

$$\frac{\partial L}{\partial p} = - \lambda MCp - \lambda C'x + \begin{Bmatrix} a_1 b_1 p_1^{-b-1} \cdot \sum_1^5 \alpha_i \\ a_2 b_2 p_2^{-b-1} \cdot \sum_2^5 \alpha_i \\ \cdot \\ \cdot \\ \cdot \\ a_5 b_5 p_5^{-b_5-1} \cdot \alpha_5 \end{Bmatrix}.$$

Gradient search methods are used in either model to find a near-zero gradient. Lagrange multiplier step size and perceived pay step size are varied separately as a function of the number of interactions and of the decrease in the length of the gradient vector. Convergence to a local optimum occurs generally within 40 iterations but is very critically dependent upon the step size selection procedure. Optimal step size algorithms are not used but could be implemented if necessary. No attempt has been made to prove or test empirically that this local is a global optimum, but our opinion is that this is a global extreme point.

Experience has indicated that convergence of the gradient search may not occur if the step size selection is not satisfactory. Computation time on Rand's IBM 370/158 took less than 12 seconds for compilation and execution.* In no case did convergence to an inflection or a minimum point occur.

The initial starting point in any gradient method is important, since the search may move toward the multiple inflection, maximum, or minimum points of the nonlinear Lagrangian. The starting point used here is found by using a perceived budget constraint rather than the correct direct expenditure budget. Hopefully, this will allow us a starting point close to the global optimum. In the optimization max $s'x$ with a perceived budget constraint $p'x \leq B$, differentiation of the

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*A program listing is available from the authors.

total Lagrangian with respect to basic decision variables p_j produces the following equations for each j:

$$a_j b_j p_j^{-b_j-1} \left[s_j - \lambda p_j + \sum_{i=j+1}^{n} r_i(p_i) \cdot r_{i-1}(p_{i-1}) \cdots r_{j+1}(p_{j+1})(s_i - \lambda p_i) \right] - \lambda = 0 \ .$$

These are a set of n nonlinear equations of degree $b_j + 1$ and can be solved recursively given each λ.

The first equation for p_n is simply

$$a_n b_n p_n^{-b_n-1}(s_n - \lambda p_n) - \lambda = 0 \ .$$

The second would be

$$a_n + b_{n-1} p^{-b_{n-1}-1}[s_{n-1} - \lambda p_{n-1} + (s_n - \lambda p_n) r_n(p_n)] - \lambda = 0 \ .$$

When solved, p_n^* and $r_n(p_n^*)$ can be substituted into the (n - 1)*st* equation. This recursion continues so that each equation contains only one decision variable and can be solved numerically using Newton's method given a specific value for the Lagrange multiplier λ.

Everett's[(24)] results assure us that if we happen to find a λ that achieves the budget constraint, then we should use it and the policy generated by i, since it is optimal (or near optimal). Fox and Landi[(25)] review methods for finding λ in the case of one constraint and suggest bisection which proceeds by successively halving an interval λ_1 and λ_2. The two starting points are chosen so that the budget generated $B(\lambda_1) < B$ and $B(\lambda_2) > B$. When the midpoint λ_k of the k*th* iteration produces a budget $B(\lambda_k)$ sufficiently close to B, the procedure stops. Bisection is mini-max in the sense that it minimizes the maximum number of iterations required to locate the root in an interval of fixed length.

A computer program has been written which calculates optimal pay policies $p^*(\lambda)$ and annual perceived budget expense $B(\lambda)$ under $p^*(\lambda)$. Bisection is used to converge to a λ^* such that $B(\lambda^*)$ is within one percent of the desired budget B. This program is used as a subroutine to obtain a good starting solution for the main program that uses the proper wage budget constraint.

Since p is perceived pay, $p'x$ is the perceived annual budget. The actual annual wage and retirement expense to the government differs from the perceived budget because of the discount rate and retirement factors.

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